The Art of Square Dancing: Math in Motion

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SENIOR HONORS THESIS

Submitted In Partial Fulfillment of Requirements of the

College Scholars Honors Program

North Central College

June 4, 2010

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The Art of Square Dancing: Math in Motion

Abstract

We will explore many ways of viewing square dancing through various mathematical lenses. Following directions from a caller, eight square dancers work together to rearrange themselves in various geometric patterns before returning to their original positions. Each square dance call produces a permutation of the dancers. We will analyze these permutations using concepts of group theory, such as the order of a call. Geometry also plays a role as the execution of a square dance call preserves symmetry and involves varying amounts of angular rotation. Recently, square dancing has been taken to new mathematical levels as callers introduce more complex ways to vary traditional calls, including ways to reconfigure standard formations making it possible for six couples to dance four-couple choreography, often called hexagon dancing. We will investigate how changing the rules of square dancing affects the algebra and geometry of the dance.

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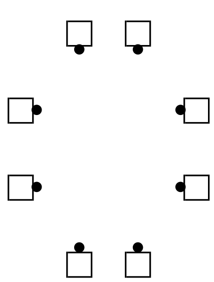
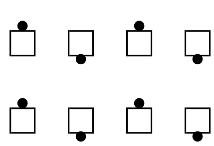
1 Square Dancing Basics

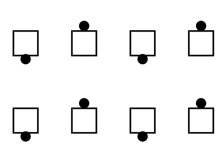
Modern square dancing comes in many forms, the most traditional of which is four-couple square dancing. In this type of square dancing, couples place themselves on the sides of an imaginary square. These couples must work together in their square, performing unrehearsed choreography, to complete the dance without breaking down. They must follow through with calls that are given to them by a caller. The caller's responsibility during the dance is to rearrange the eight individuals in various formations and then to untangle them again in order to return the square to the original starting positions, called "home". This process is much like the scrambling and unscrambling of a Rubik's cube.

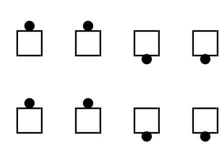
There are thousands of different square dancing calls, but only about 1,000 are used regularly in modern square dancing. Each call has its own name and a series of movements associated with it. Furthermore, there are about 100 concepts, which modify these calls. Concepts generally precede a call and give more specific information regarding what to do with the call. For example, the concepts may tell the dancers to complete only part of a call, to start with a different hand, or to dance with other individuals during that call than may have been expected.

There are many different types of formations seen in a square dance. Each of these formations has a different list of calls that can be performed from that formation. Some calls are more universal than others, meaning that they can be called from many formations, when other calls are specific to a particular formation. Some formations include: the squared set, which looks like a square and where dancers begin and end each dance; right-handed (and left-handed) parallel waves, which are called waves because of their shape (with dancers alternating facing directions); different types of lines (with dancers facing different directions depending on which type of line formation); and many others, some of which can be seen below. In the diagrams, each box represents a dancer and the dot on one side of that dancer indicates which direction the dancer is facing. For example, in the first diagram of a squared set, the dancers are placed on an imaginary square, with two dancers on each side, and all of the dancers facing the center of the square.


squared set

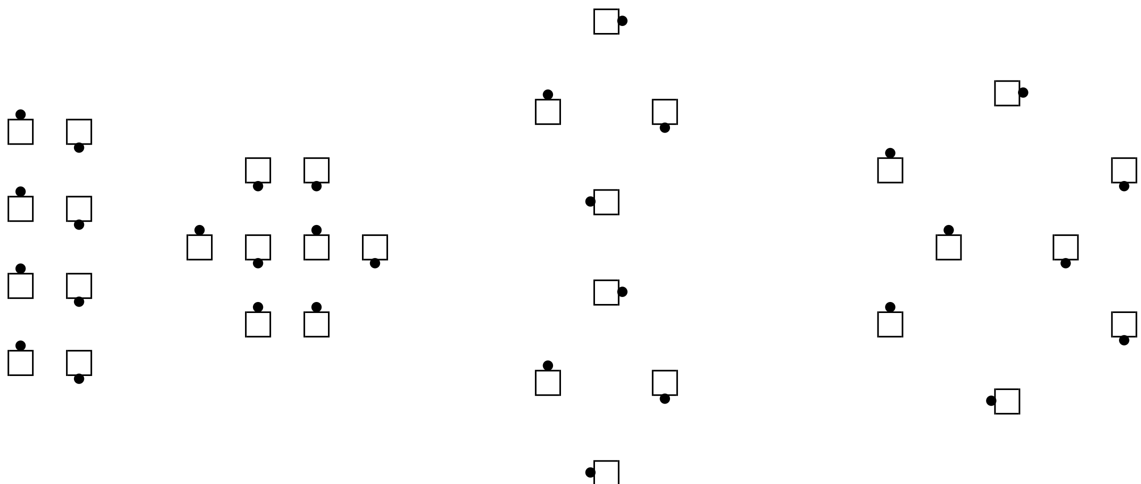

right-handed parallel waves


left-handed parallel waves


right-handed two-faced lines

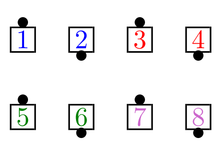
Note the difference in right-handed and left-handed waves. As dancers will be holding hands within each wave, the "handed-ness" is determined by the handhold of the two dancers on the end of each wave (not the two center dancers). In other words, if the two dancers on the end of the wave are holding right hands, then the wave is called a right-handed wave, and similarly for left-handed waves.

Below are some of the more interesting formations seen in square dancing:



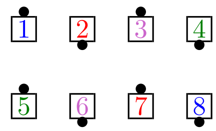
column quarter tag diamonds hourglass

During a square dance, dancers must be aware of two individuals: their partner, who changes throughout the dance, and their opposite, who is always their opposite. A square dance partner is not the person with whom a dancer attends a square dance, although at particular moments throughout the dance, this person may in fact be his/her partner. A square dance partner changes from call to call, and often during the course of a call as well. A dancer's partner will either be to his left or to his right. In order to figure out who a dancer's partner is, we can divide the formation into pairs, and each pair will represent a dancer with his/her partner. Using the right-handed parallel wave formation, the figure below presents pairs of dancers with their partners in the same color.



dancers and their partners

The other person a dancer must be aware of is his opposite. This person is the dancer's mirror image across the center point of the formation. If the formation were to rotate 180°, these two dancers would be in the same position. Being aware of the opposite is important because a dancer and his/her opposite will always perform the same movements of a call, just moving in opposite directions, so as to remain mirror images of each other. We will call this property ***square dance symmetry***. A dancer's opposite remains his/her opposite for the entire dance. Again, the figure below presents the pairs of dancers with their opposites in the same color.



dancers and their opposites

But square dancing did not begin with the complex call structure that we see here. Square dancing evolved from American and European folk dancing, where less emphasis was placed on memorization and learning the definitions of calls. At first, square dancing consisted mainly of short, repetitive sequences of calls with simple choreography that could be quickly taught and learned. Square dancing has developed over the years into a more complex and mentally stimulating experience. Calls with specific instructions were introduced and the repetitive nature of the dance was eliminated, resulting in more interesting traffic patterns for the dancers and a grander flow of the dance. Today, all dancers must learn the various definitions of calls, recorded by a national organization called Callerlab, and must listen carefully to the caller to execute the calls with precision [3].

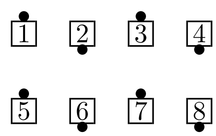
As square dancing became standardized across the nation, and eventually, throughout the world, it became easier for dancers to learn calls. Instead of learning different calls for the different regions of the world, dancers could focus on one set of definitions, found in Bill Burleson's *Square Dance Encyclopedia*, published in 1970, and later in Callerlab's lists of definitions [9]. Callers began to write calls and add to these square dance dictionaries. Lee Kopman, for instance, "has invented more calls in popular usage than any other caller in the history of Square Dancing" [11]. Most of these calls are more mathematical than the previous ones, with more intricate choreography and complex movements. Square dancing has evolved into a puzzle-solving activity, a dance with various mathematical twists, such as permuting parts of a call, adding or deleting angular rotations, and introducing concepts that "link" dancers together or involve reflections. Many of these mathematical ideas in square dancing have been studied in recent years by callers who want to understand the mathematical nature that has been infused into the dance (see [1], [2], [7], [13], and [14] for examples of such papers). In furthering this understanding, we will attempt to find new mathematical patterns and connections in what has mathematically evolved to become modern square dancing.

2 A Square Dancing Group

Understanding the basic ideas of square dancing, we can begin to analyze the more mathematical properties of the dance. We will do this by considering sets of square dancing calls with a certain similar property.

Let $\mathcal{F}$ be a square dance formation and $G_{\mathcal{F}}$ be the set of all finite sequences of calls that take an arrangement of dancers in $\mathcal{F}$ to another arrangement of dancers in $\mathcal{F}$ (maintaining square dance symmetry). We will refer to these sequences of calls as ***queues*** (of calls) and suppose that the starting arrangements for each of these queues is the same, so as to compare the ending positions they produce.

For example, if considering right-handed parallel waves as the formation, then we will assume all queues to start from the arrangement shown below and will consider the ending arrangements of the dancers.



beginning arrangement of dancers

Consider two queues in $G_{\mathcal{F}}$ (both with the same starting arrangement as explained above), q_1 and q_2. We define a relation $\sim$ such that $q_1 \sim q_2$ if the ending arrangements of dancers resulting from q_1 and q_2 are the same. This will essentially ignore the traffic patterns of the queues, by associating two queues if they induce the same permutations of dancers within $\mathcal{F}$, and not caring *how* the dancers move to these positions.

We see that $\sim$ is an equivalence relation. This means that the relation meets three requirements.

1. Reflexive: The relation $\sim$ is reflexive, meaning a queue q_1 will induce the same ending arrangement as itself.

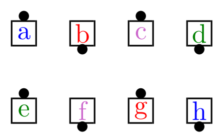
2. Symmetric: The relation $\sim$ is symmetric, meaning that if a queue q_1 and a queue q_2 induce the same permutation of dancers, then the queue q_2 and the queue q_1 induce the same permutation of dancers as well. (These queues are related to each other.)

3. Transitive: The relation $\sim$ is transitive. This means that if q_1 and q_2 induce the same ending arrangements (and are thus related to each other), and q_2 and q_3 are related to each other, then q_1 and q_3 must also be related to each other.

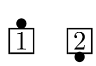
So now we have a way of relating queues to each other depending on the starting and ending arrangements of dancers within a formation. We can categorize all of the queues in $G_{\mathcal{F}}$ by putting them in subsets, where each subset contains queues that induce the same permutations of dancers in $\mathcal{F}$. These subsets are called equivalence classes. We can reduce the number of queues in $G_{\mathcal{F}}$ by considering only these equivalence classes, and we will name this new set $G_{\mathcal{F}}/\sim$. This notation indicates that we will take all of the queues in $G_{\mathcal{F}}$ and sort them into their equivalence classes under the $\sim$ relation, so that we have only one equivalence class for every permutation of dancers.

Remark: $G_{\mathcal{F}}/\sim$ is the entire set of square dance permutations, meaning every possible ending arrangement of dancers appears as an outcome of exactly one equivalence class of queues.

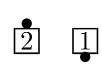
Using right-handed parallel waves and the diagram below, we see that this is true.



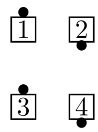
We can manipulate our dancers through a series of calls named *Trade* and *Box Circulate*. A *Trade* requires two adjacent dancers to switch places, while a *Box Circulate* requires four dancers to become a part of a cycle, each dancer moving to the next position clockwise (or counterclockwise) in the cycle. By using queues solely consisting of *Trade*s and *Box Circulate*s, we can rearrange the dancers into every possible ending position (following the rules of square dance symmetry).



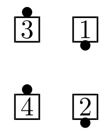
before *Trade*



after *Trade*



before *Box Circulate*



after *Box Circulate*

First, we will find a queue that moves the appropriate two dancers to positions a and h; this is possible, since for now we do not care where the other six dancers end up. We will keep the two dancers that we have placed in positions a and h where they are and will work to "unscramble" the remaining six dancers. If the dancers in positions d and e are not in the desired end positions, then we will first find the correct dancers and use a queue of *Trade*s and/or *Box Circulate*s to move these dancers to positions c and f, and then another *Trade* to leave dancers in the desired positions d and e. Now the only dancers who may not be in the desired end positions are the dancers occupying positions b, c, f, and g, and again using a queue of *Trade*s and *Box Circulate*s, we can move dancers to the correct positions. Thus we have the entire set of square dance symmetric permutations of dancers.

$G_{\mathcal{F}}/\sim$ with the operation of queue composition, meaning performing one queue after another, is a ***group***. In order to be a group, a set must meet four requirements.

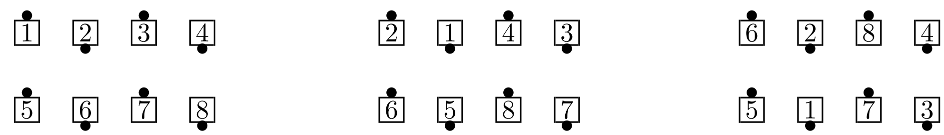
1. Closure: If you take a queue and compose it with another queue (meaning complete one queue followed by the other queue), then we will still have a queue of calls.

2. Associativity: Consider three queues, q_1, q_2, and q_3. If we perform them in such a way that we complete queues q_1 and q_2 in one sweeping movement, followed by q_3, then we will have the same ending arrangement of dancers as if we first completed queue q_1, followed by q_2 and q_3 in one movement. This is true since every part of every queue is being performed, regardless if we pause after one of them or not.

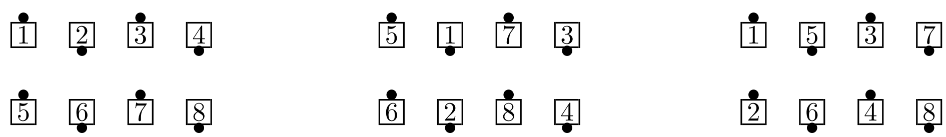
3. Identity: A group must have an identity element. In terms of square dancing, this means that there must be a queue q_e such that if you perform any other queue q_1, followed by q_e, or if you perform q_e first, followed by q_1, the ending arrangement will always be the same as q_1. This identity queue, q_e, is simply for dancers to do nothing and not move.

4. Inverses: Each queue of calls must have an inverse, or another queue that will undo the permutation, essentially reversing the arrangement of dancers. In other words, if a queue q_1 has starting arrangement a and ending arrangement b, then the inverse must have starting arrangement b and ending arrangement a. We know that every queue has an inverse since our set $G_{\mathcal{F}}/\sim$ is the entire set of symmetric permutations.

It is interesting to note that $G_{\mathcal{F}}/\sim$ is not abelian, meaning that the ending arrangment of dancers may change depending on which of two queues is performed first. For example, completing a *Trade* followed by a *Split Circulate* is different from first doing a *Split Circulate* and then a *Trade*. The ending arrangements can be seen below.



beginning arrangement — after a *Trade* — after a *Split Circulate*



beginning arrangement — after a *Split Circulate* — after a *Trade*

Notice that the beginning arrangements of dancers in each of these is the same, but the ending arrangements of dancers are not the same. Thus we have shown that $G_{\mathcal{F}}/\sim$ is a group, but is not an abelian one. We will call this the ***Square Dancing Group relative to*** $\mathcal{F}$, and it will lead us to the next section, focusing on a particular property of groups.

3 Possible Orders of a Square Dance Call

Now that we have a square dance group, we can use concepts from group theory, like that of "order", to explore the complexity of our group. More precisely, if G is a group with identity element e and binary operation $*$, then the ***order*** of an element $a \in G$ is the smallest positive integer k such that $a^k = e$ [12]. In terms of square dancing, we will take a beginning formation (right-handed parallel waves) and determine the fewest number of times a queue of calls must be performed repeatedly until dancers are in the same position of the formation in which they started. By abuse of notation, from now on we will refer to all queues of calls simply as calls. The order of a square dancing call, then, will only exist if the call can be performed multiple times. We will thus consider only the calls that take dancers from the right-handed parallel wave formation to another right-handed parallel wave formation.

3.1 Without Rotation in Formations

We begin by considering the following starting and ending formations, where dancers move from their starting formation to a similar ending formation, with potentially different places in the formation.

We wish to consider all the possible orders of a square dancing call. That is, we want to find the number of times the same call must be iterated for dancers to return to their original positions. Since we are using the square dancing group, as described in the previous section, we can borrow a theorem from group theory, namely Lagrange's Theorem. More specifically, we want to use a consequence of this theorem:

Theorem 1 (Corollary to Lagrange's Theorem)

Let G be a group with N elements. If $a \in G$, then the order of a divides N. (i.e. The order of any element of G must divide the order of G.)

In other words, the order of a single square dancing call must divide the total number of potential arrangements of dancers within the starting formation for that call. We must first find this number of arrangements of dancers in our square dancing group.

The first dancer has 8 positions to choose from. His/her opposite has no choice of which position to take, but must do the "mirror-image movement" of the first dancer, since square dancing calls are symmetric, and thus this dancer will end in the opposite position. Then there are 6 positions left to choose from for the next dancer, and once again his/her opposite has no choice of position. The next dancer will have only 4 positions to pick from, with his/her opposite moving to the corresponding opposite position, and the last two dancers have just 2 positions left to fill. Thus there are $8 \cdot 6 \cdot 4 \cdot 2 = 384$ possible arrangements of dancers in the formation.

Theorem 2 (A Consequence of Lagrange's Theorem for the Square Dancing Group)

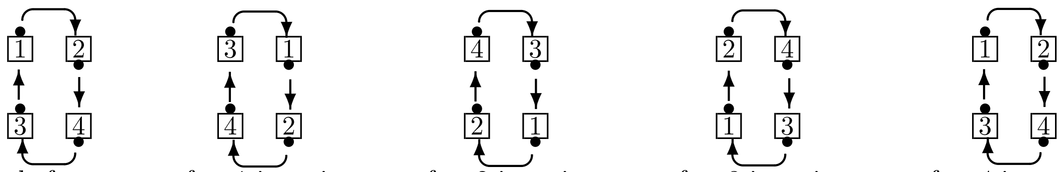
The order of any single square dancing call must divide 384.

Then the order of any square dancing call must be 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 128, 192, or 384. However, this list of possible orders of a call can be narrowed down, and that is the goal for the next part of this section. In order to see which orders of a call are possible, we will look at the potential orders of a call under certain circumstances, by fixing dancers. If a dancer is fixed, then the dancer will start and end in the same position, and so once the call is completed, it will appear as if the dancer had not moved at all. Also, since square dancing calls are symmetrical, the dancer's opposite will be fixed as well. We can then consider fixing all dancers, six dancers, four dancers, two dancers, and no dancers, and look at the effect of this constraint on the possible orders of a square dance call. However, before looking at the individual cases, we must first understand the meaning of a cycle.

Definition 1 (An n-Cycle of Dancers)

*Let $a_1, a_2, a_3, \cdots, a_n$ be distinct dancers in the formation. Then a **cycle of length n**, or an **n-cycle**, is the rotation of these n dancers by one position. (i.e. $a_1 \to a_2$, $a_2 \to a_3$, $\cdots$, $a_{n-1} \to a_n$, $a_n \to a_1$.)*

For example, a 2-cycle would consist of two dancers switching places (and would have order 2), while a 4-cycle would involve four dancers rotating one position at a time (and would have order 4). A 4-cycle can be seen below:



before   after 1 iteration   after 2 iterations   after 3 iterations   after 4 iterations

Similarly, the order of any n-cycle is n because each dancer will need to rotate into all other positions (there are $n-1$ of them, not including the dancer's original position) before returning to where they started. In square dancing, since there are eight dancers, we can think of having n-cycles where $n = 1, 2, \cdots, 8$. Note that a 1-cycle indicates a dancer changing positions with himself, and thus this dancer is simply a fixed dancer.

Now, using n-cycles, we can determine the possible orders of a call under the restriction of fixing dancers.

<u>Case 1:</u> (all eight dancers are fixed)

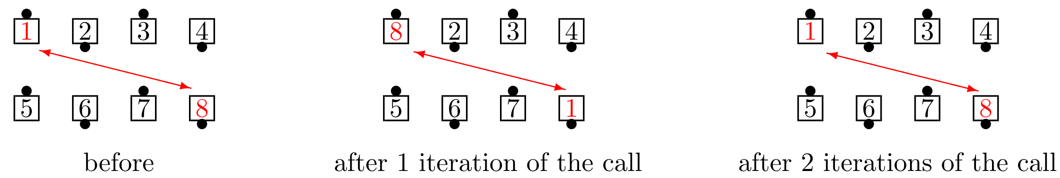
In this case, all eight dancers start and end in the same positions of the formation. Then the order of a call that fixes eight dancers must be 1, since the call only needs to be performed one time for dancers to return to their original positions, as seen below.

before   after 1 iteration of the call

<u>Case 2:</u> (exactly six dancers are fixed)

With exactly six dancers fixed, the only dancers not ending in their original positions would be one dancer and his/her opposite. These two dancers must switch places, resulting in a 2-cycle, and so the call will need to be performed two times for these dancers to return to their original positions. In this case, the square dancing call must have order 2.

For example, if dancers #1 and #8 are not fixed, the call's positions will be the following:

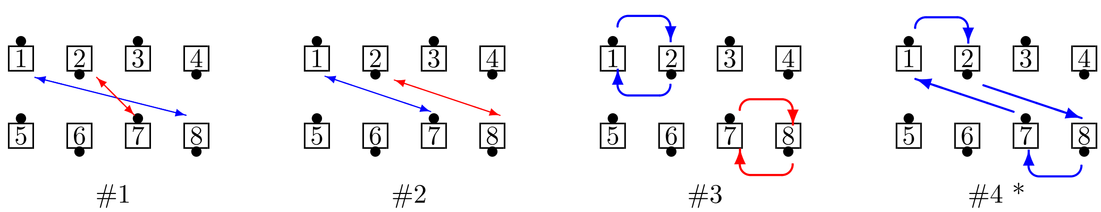


We see that after completing the call two times, every dancer has returned to his/her original position of the formation, and thus the order of the call is 2.

<u>Case 3:</u> (exactly four dancers are fixed)

With exactly four dancers fixed, there are five distinct possibilities of the ways in which the dancers may change places in their formation.

For example, with dancers #3, #4, #5, and #6 fixed, the possibilities are:



(*) and the reverse direction

- In calls #1, 2, and 3, there are two disjoint 2-cycles of dancers. As in the last case, this will result in the call needing to be iterated twice for dancers to return to their original positions. Thus the order of a call under this condition is 2.

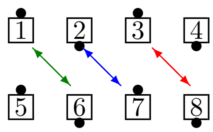
- In call #4 (and its reverse direction), dancers form a 4-cycle. Thus the order of this type of call is 4.

Therefore if exactly four dancers are fixed, a call must have order 2 or order 4.

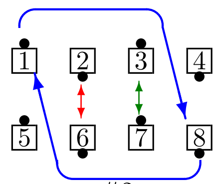
Case 4: (exactly two dancers are fixed)

With exactly two dancers fixed, there are twenty-nine distinct ways in which the dancers may change positions in the formation.

For example, with dancers #4 and #5 fixed, the possibilities are:



#1



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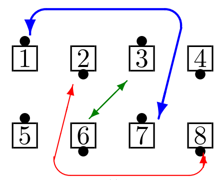

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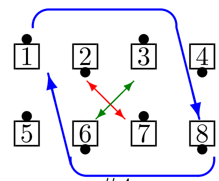

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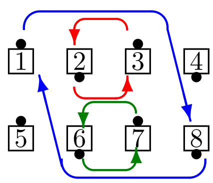

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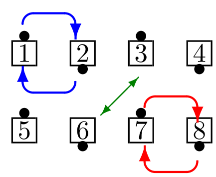

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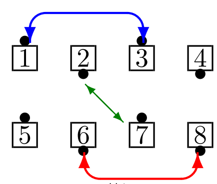

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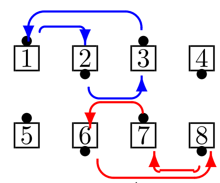

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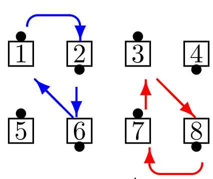

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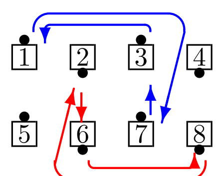

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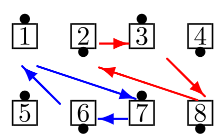
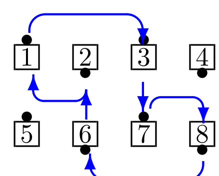


#11 *



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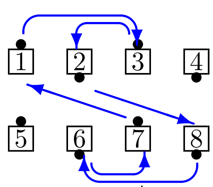

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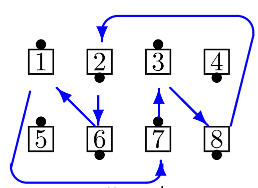

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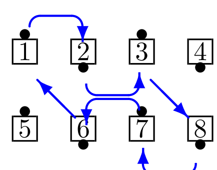

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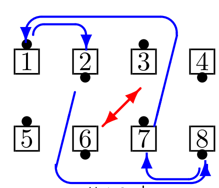
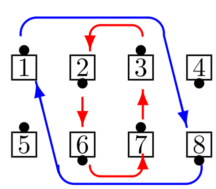
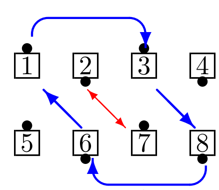


#16 *



#17 *



#18 *

(*) and the reverse direction

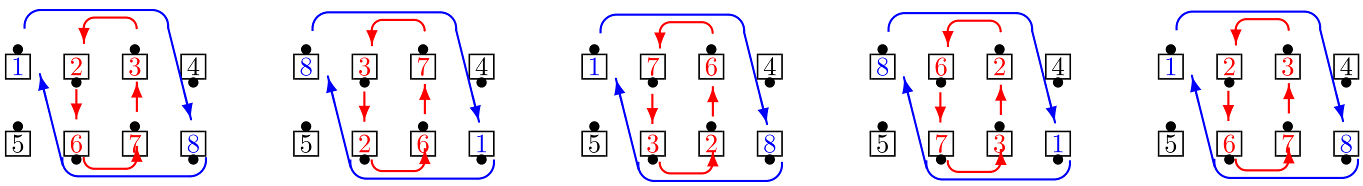
- In calls #1 through 7, the six moving dancers form three disjoint 2-cycles. For these calls, just as in the previous cases, the order is 2.
- As seen in calls #8 through 11, the six dancers form two disjoint 3-cycles. In this case, each dancer in the group of three will need to end in both of the other two positions before he can return to his starting position. Thus the call will need to be performed three times for each dancer to return to his/her original position, giving an order of 3.
- Calls #12 through 15 have all dancers working together in a 6-cycle. Thus the order of these calls is 6.
- Finally, in calls #16 through 18, the six dancers have split into two disjoint cycles. One of these cycles has length 2, while the other has length 4. We have seen both of these types of cycles in Case 3. The 2-cycle will need to complete the call twice in order for these two dancers to return to their starting positions, while the 4-cycle will need to complete the call four times in order for the dancers to return to their beginning positions. Since the call must have a single order, we must compare the "mini-orders" for each cycle of dancers. One cycle has mini-order 2, while the other has mini-order 4. We can see that if we perform the call two times, only two out of the six dancers will be back to where they started. On the other hand, if we perform the call four times, every dancer will be back to their original position, since the 4-cycle has order 4 on its own, and the 2-cycle will have returned after the second iteration and again after the fourth.

This leads us to the following theorem:

Theorem 3

The order of a square dancing call is the least common multiple of the "mini-orders" of the disjoint cycles of dancers in that call. (e.g. If the call involves exactly two disjoint cycles of dancers where the mini-order of one cycle is a *and the mini-order of the other cycle of dancers is* b*, then the order of the call is* $\mathrm{lcm}(a, b)$*. This same principle will work for more than two disjoint cycles as well.)*

For example, consider call #17 from the list of possible movements. The dancers' positions will be the following:



Since this call has exactly two disjoint cycles of dancers, one with mini-order 4 and one with mini-order 2, the call has order $\text{lcm}(4, 2) = 4$.

Therefore, with exactly two dancers fixed, a square dancing call can have order 2, 3, 4, or 6.

<u>Case 5:</u> (no dancers are fixed)

This case incorporates parts of each of the previous cases, and looks first and foremost at the maximum possible order of a square dance call, which occurs when there are no fixed dancers.

Theorem 4 (Maximum Possible Order)

The maximum possible order of a square dance call starting and ending in $\mathcal{F}$ (preserving square dance symmetry) is 8.

Proof

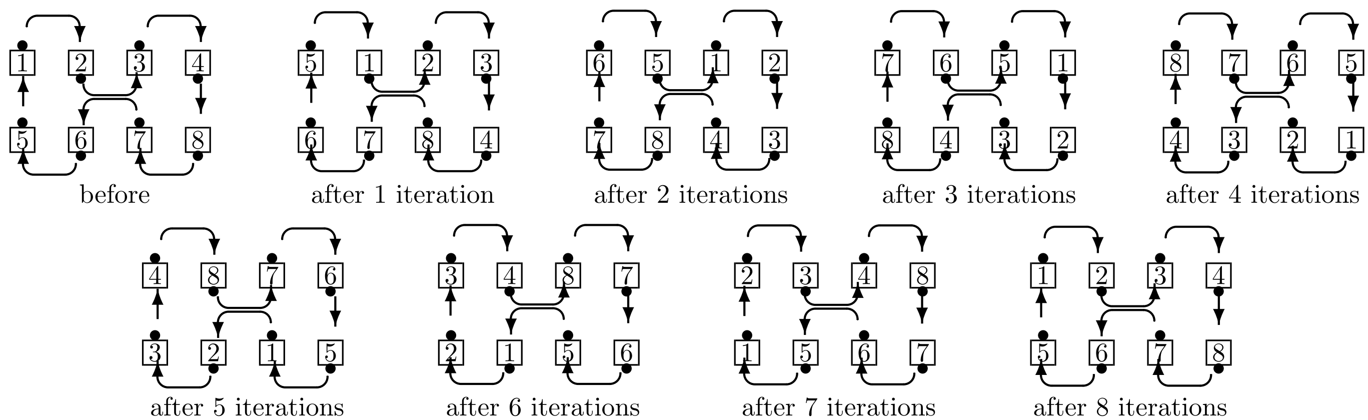
Suppose at least one dancer is fixed. Then, by the cases above, the maximum possible order of a square dancing call is 6. However, we claim that the maximum possible order is 8, and thus it must be the case that no dancers are fixed.

Then, each dancer must end in a different position every time the call is performed (in order for dancers not to be fixed). Note that there are only six ways to partition all the dancers into disjoint cycles of length greater than 1 (see Remark below):

- four 2-cycles
- two 2-cycles and one 4-cycle
- two 4-cycles
- one 2-cycle and two 3-cycles
- one 2-cycle and one 6-cycle
- one 8-cycle

Using Theorem 3, we see that the maximum possible order is 8.

It is important to note that it is possible to have this order. By imagining the formation as a 2×4 grid, we can simply ask the dancers to move to the next position clockwise in the grid. A particular call with this property is *Inroll Circulate*.



This call guarantees that dancers will end in all positions before arriving back at their original places, giving the maximum order of 8. $\square$

<u>Remark:</u> We see that if n is even, an n-cycle is possible, since it could include both a particular dancer and his/her opposite. If n is odd, it is only possible to have n-cycles if there are an even number of these n-cycles. For example, since 3 is odd, we can only have 3-cycles if we have an even number of 3-cycles, since one dancer will be in the first 3-cycle and his/her opposite will be in the second 3-cycle. For this reason, we cannot have 5-cycles and 7-cycles, since it is not possible to have an even number of these cycles with only eight dancers.

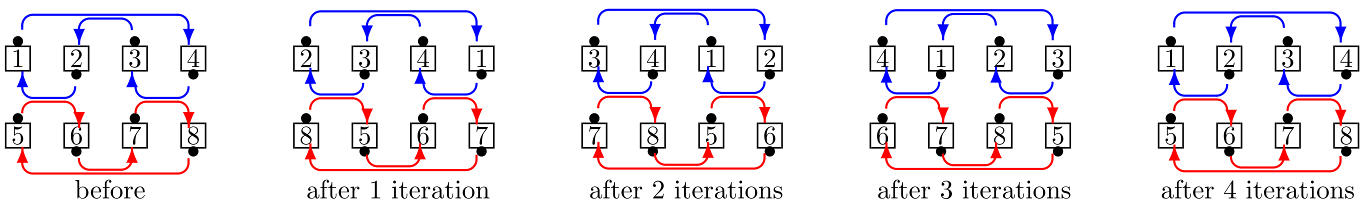
Now we have a cap for possible orders to consider under the case where no dancers are fixed. This narrows down the orders that are left to consider for this case to: 1, 2, 3, 4, and 6, according to the statement after Theorem 2.

- The first order is trivial. It is not possible to have order 1 since that would require all dancers to be fixed, and we want none of them to be fixed.

- For order 2, we can ask every dancer to *Trade* with his/her partner. These two dancers will still be partners after completing a *Trade*, and so the order is 2.

- With no dancers fixed, the order cannot be 3. This order is only possible with two disjoint 3-cycles, but that would mean leaving two dancers fixed. If these dancers were to trade places, then the call would no longer have order 3, but 6, since the order would be $\mathrm{lcm}(3, 2) = 6$.

- For order 4, we can split the dancers into two 4-cycles. For example, each wave can be its own cycle. By asking dancers to move to the next position of the wave, all moving in the same direction (and the dancer on the end can run to the first position in the wave), all eight dancers will be involved, but each wave will have mini-order 4, giving a final order of 4 as well. A call with this property is *Outroll Circulate*, seen below.



- For order 6, we can use a 6-cycle from the previous case (with 2 fixed dancers), but ask the dancers who were fixed in that call to trade places with each other instead. Then the mini-orders of our two disjoint cycles are 2 and 6, giving a final order of $\mathrm{lcm}(2, 6) = 6$.

In General:

To summarize Cases 1 through 5, we see that the allowable orders of a square dancing call depend on how many dancers are fixed. The following table shows the possible orders for each number of fixed dancers.

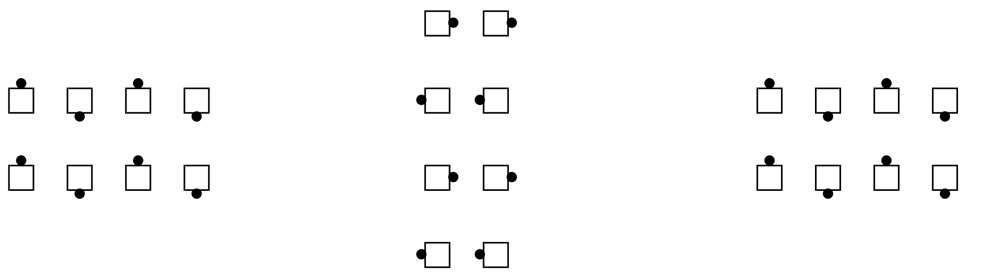
Number of Fixed Dancers	Possible Orders of a Call
8	1
6	2
4	2, 4
2	2, 3, 4, 6
0	2, 4, 6, 8

Notice that every order less than or equal to 8 shows up in the table, except for 5 and 7. These two orders do not appear since they would violate the square dance symmetry (and do not divide 384, our number of potential arrangements of dancers).

Also, these orders were obtained by considering the right-handed parallel wave formation. This can be extended to any formation, as we considered only the fixed positions of the dancers, and not the traffic patterns of the calls. Since we only kept track of which dancers belong to each cycle, and not the dancers' positions within the cycle, the possible call orders relative to any formation $\mathcal{F}$ will be the same.

3.2 With Rotation in Formations

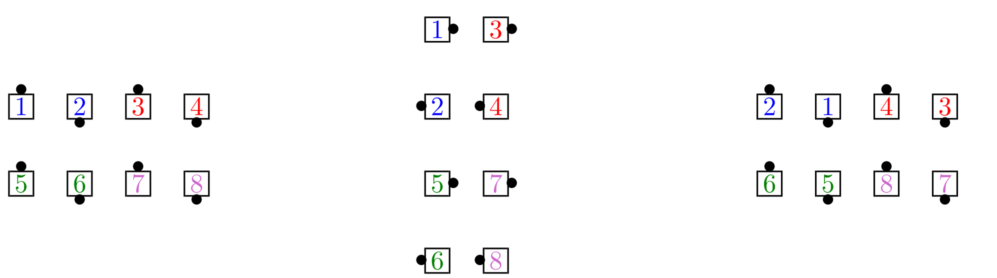
Often times a square dance call will take a starting formation and change it to a different ending formation. Of course it is not always possible to find the order of these calls because in most cases, the call cannot be performed a second time from the new starting formation. There is a certain type of call, however, that changes the ending formation, but in a way that the call can always be performed again. Next we will consider this type of square dancing call that adds a rotation to the starting formation. After each iteration of the call, the formation will rotate 90° in the clockwise direction.



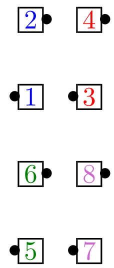
before after 1 iteration of a call after 2 iterations of a call

Since the order of a call is determined by dancers beginning and ending in the same positions of the same formation, the order of a call that undergoes this type of rotation must be even.

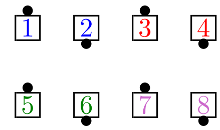
For example, a call that has this type of rotation is called *Hinge*. A *Hinge* is performed by walking in a quarter circle towards your partner, using the handhold as a pivot. This particular call has order 4, as seen below.



before after 1 iteration of a call after 2 iterations of a call



after 3 iterations of a call

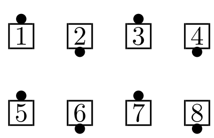


after 4 iterations of a call

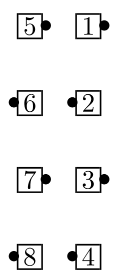
To find the possible orders of a square dancing call involving the rotation in formation, we will use the same method as before. We can more easily determine the possible orders by fixing dancers.

Case 1: (all eight dancers are fixed)

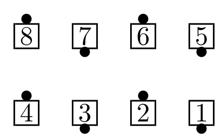
Fixing all eight dancers requires dancers to start and end in the same positions. But adding a 90° rotation will affect the order of a call, since dancers would end in the same positions *before* the rotation. This means that upon the completion of every iteration of the call, the entire formation will be picked up, rotated, and put back down, as if the formation had been on a rotating platform while dancers remain motionless. This can be seen below:



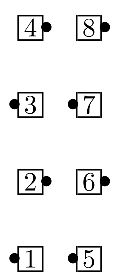
before



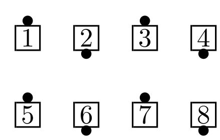
after 1 iteration of a call



after 2 iterations of a call



after 3 iterations of a call



after 4 iterations of a call

Thus the order of this type of call, with eight dancers fixed, is 4.

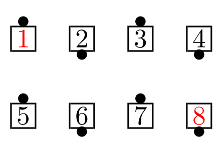
We can generalize this to other types of calls as well. In the previous example, we can look at dancers #1 and #8, and temporarily ignore the other six dancers. We see that it takes four iterations of the call for these two dancers to return to their original positions, regardless of what the other dancers are doing during the call. In light of Theorem 3, this leads us to the following theorem:

Theorem 5

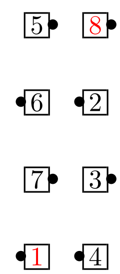
If a call (under rotation in formation) fixes any two dancers, then that call must have an order divisible by 4.

Case 2: (exactly six dancers are fixed)

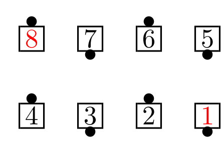
Since at least two dancers are fixed, by Theorem 5, the order of a call with six dancers fixed must be divisible by 4. As seen before, six dancers fixed leaves only two dancers (a dancer and his/her opposite) to end in a different corresponding position of the now rotated formation. So this dancer and opposite must trade places in the formation. We can examine this using the diagrams, as before. We will assume that dancers #1 and #8 are not fixed.



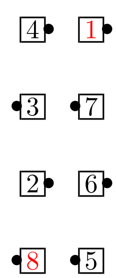
before



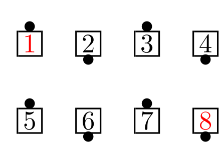
after 1 iteration of a call



after 2 iterations of a call



after 3 iterations of a call



after 4 iterations of a call

Again, we see that this type of call has order 4. Since all dancers are fixed, except one dancer and his/her opposite, this leads us to another generalization, again using Theorem 3.

Theorem 6

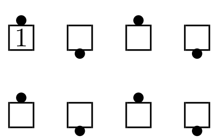
If a call (under rotation in formation) requires any dancer and his/her opposite to trade places, then that call must have an order divisible by 4.

In fact, this can be generalized even further. The theorem not only applies to a dancer and his opposite trading places, it can be applied to any two dancers trading places as well.

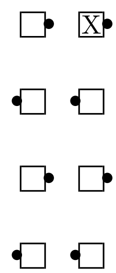
Theorem 7

If a call (under rotation in formation) requires any two dancers to trade places, then that call must have an order divisible by 4.

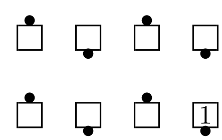
To see that this is true, consider any 2-cycle involving dancer #1. Then, only looking at the positions of dancer #1, we have:



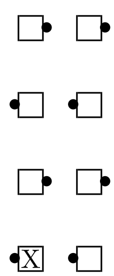
before



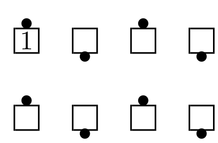
after 1 iteration of a call



after 2 iterations of a call



after 3 iterations of a call



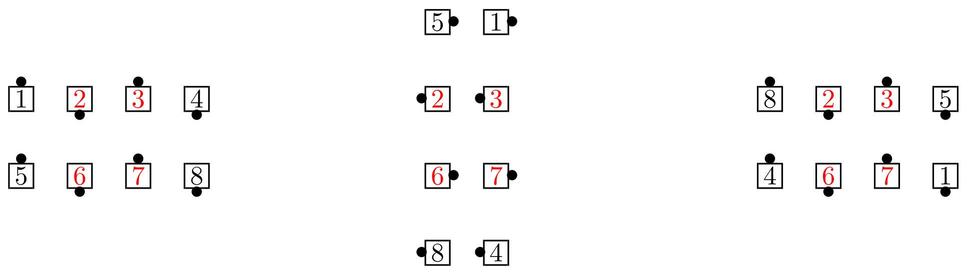
after 4 iterations of a call

where X represents dancer #1's original position, and thus the position that the dancer trading places with dancer #1 would occupy. We see that after performing the call four times, both dancers in the 2-cycle will return to their original positions, giving an order that is a multiple of 4.

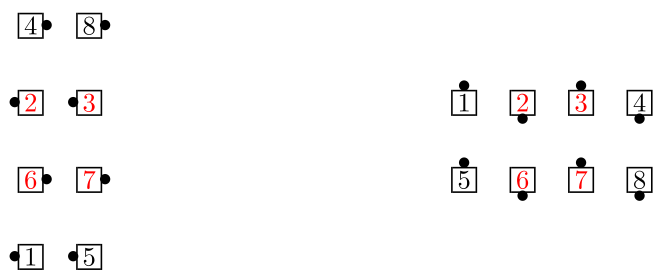
Case 3: (exactly four dancers are fixed)

As in the previous section, if exactly four dancers are fixed, then the remaining dancers must form a 4-cycle, or must break into two disjoint 2-cycles of dancers.

- If the non-fixed dancers form a 4-cycle, then the order of the call is 4 since it will take these dancers four iterations of the call in order to return to their starting positions normally, without the rotation, and after four iterations, the formation would have rotated four times, back to the original footprints. Actually, the 4-cycle has mini-order 2 with this rotation, but since there are fixed dancers, by Theorem 5, the order must be divisible by 4. This can be seen below with dancers #1, #4, #5, and #8 fixed:



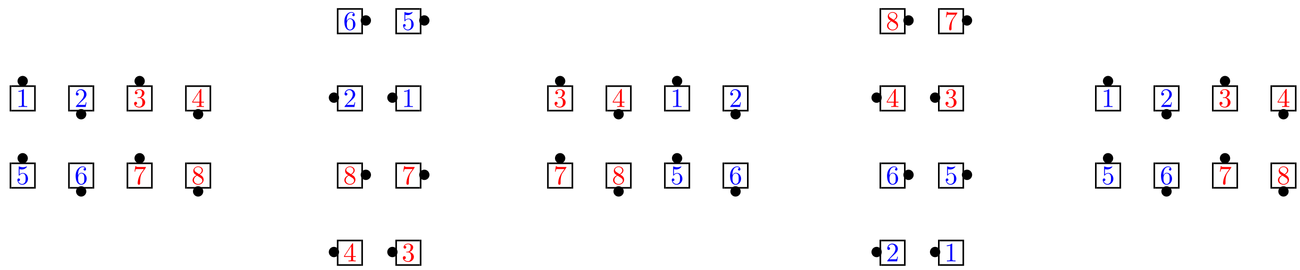
before　　after 1 iteration of a call　　after 2 iterations of a call



after 3 iterations of a call　　after 4 iterations of a call

Remark: In certain circumstances a 4-cycle may have mini-order 4 when the formation is rotated. This occurs when at least one dancer and his/her opposite are in separate 4-cycles, something that is impossible when the call involves only one 4-cycle.

In this case, the permutations may look like this:



- If the non-fixed dancers form two disjoint 2-cycles, then by Theorem 7, the call must have an order that is divisible by 4. Since there are no other cycles of dancers (other than cycles of length 1), by Theorem 3, the order actually is 4.

Thus if exactly four dancers are fixed, the order of a call must be 4.

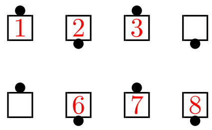
Case 4: (exactly two dancers are fixed)

With exactly two dancers fixed, there are four ways the remaining dancers can form cycles.

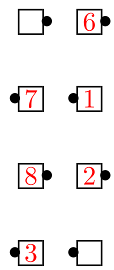
- The first of these is for the six remaining dancers to form three disjoint 2-cycles. Just as in the last case, this results in order 4.

- Another possibility is for the moving dancers to split into two disjoint cycles - one of length 4, and one of length 2. From the last case, we know that, because there is exactly one 4-cycle, it has mini-order 2 with the rotation and the 2-cycle has mini-order 4. Thus this type of call has order $\text{lcm}(2, 4) = 4$.

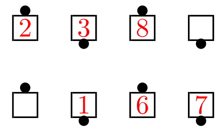
- The next possibility is for all six moving dancers to form a 6-cycle. Normally, a 6-cycle would take six iterations for all dancers to return to where they started. When we begin to rotate the formation, the call will need to be performed twelve times for all dancers in the 6-cycle to return to their original footprints. The following diagrams illustrate the effects of the rotation on a 6-cycle, ignoring the dancers in the other two positions.



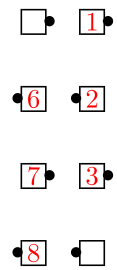
before



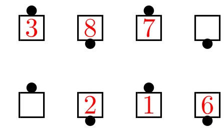
after 1 iteration



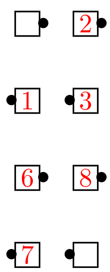
after 2 iterations



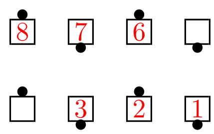
after 3 iterations



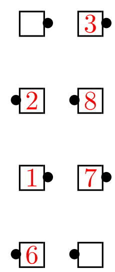
after 4 iterations



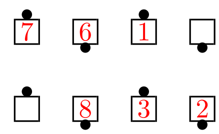
after 5 iterations



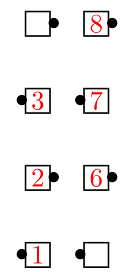
after 6 iterations



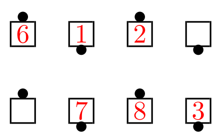
after 7 iterations



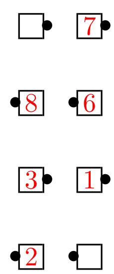
after 8 iterations



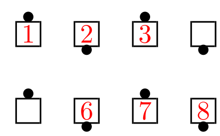
after 9 iterations



after 10 iterations



after 11 iterations



after 12 iterations

We see that a 6-cycle has order 12 with the rotation in formation. But we must also consider the dancers who are fixed. By Theorem 5, we know that these dancers have mini-order 4. Then the call must have order $\text{lcm}(12, 4) = 12$.

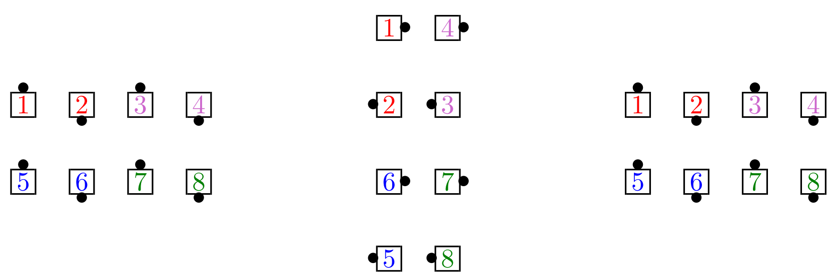
- Similarly, if the moving dancers split up into two disjoint cycles of length 3, the call will have order 12. The 3-cycles will have mini-order 12 because it normally would take the dancers three iterations to return to their starting places. But with the rotation in formation, ending on an odd number of iterations would result in a perpendicular end formation. Then our mini-order must be an even multiple of 3. Since six iterations would result in dancers ending in their opposites' positions, the mini-order of a 3-cycle must be 12, resulting from the 3-cycle going around four times. Again because of the fixed dancers, we have order $\text{lcm}(12, 4) = 12$.

Thus if exactly two dancers are fixed, a square dance call may have order 4 or 12.

Case 5: (no dancers are fixed)

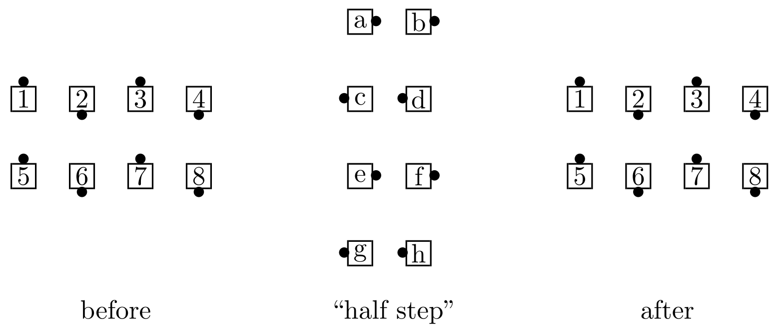
Now we will assume that no dancers are fixed. Then there are several ways in which dancers can form cycles.

- The first of these is for dancers to get into four disjoint 2-cycles. Just as before, this type of call will have order 4.
- The dancers may also split into two disjoint 4-cycles. If one 4-cycle consists of two pairs of opposites, then so does the other 4-cycle, and from the previous case, we know that each 4-cycle has mini-order 2. So by Theorem 3 the call has order 2 as well. If, on the other hand, at least one dancer and his/her opposite are in separate 4-cycles, then according to the Remark in Case 3, each 4-cycle has mini-order 4. So by Theorem 3, the call has order 4 also.
- If dancers form a 6-cycle and the other two dancers trade places, then the call will have order 12 since the 6-cycle has mini-order 12, the 2-cycle has mini-order 4, and $\operatorname{lcm}(12, 4) = 12$.
- Similarly, if the dancers form two disjoint 3-cycles and one 2-cycle, the call will have order 12.
- Another possibility is for all dancers to remain in an 8-cycle. These dancers will normally have completed the call after 8 iterations, and since there is a full set of rotations after every 4 iterations of the call, the order will still be 8 with the rotation.
- Because of the added rotation, there is potentially another set of calls having order 2. (We have seen that this can occur when there are two disjoint 4-cycles each consisting of two pairs of opposites; however, we want to find all possibilities in which this actually happens.) It turns out that there are twelve combinations of these calls with order 2. For example, the following call will have order 2: the end dancer facing out and his partner will do a *Hinge*, while the end dancer facing in and his partner will do a *Cast-Off* $\frac{3}{4}$ (three successive *Hinge*s).



before after 1 iteration after 2 iterations

We will call the middle formation, which consists of the vertical orientation of the right-handed parallel waves, the "half step" since it provides the halfway mark between the horizontal orientations of our beginning and ending formations in a call that has order 2. In fact, there are exactly twelve of these "half steps". Let's see why:



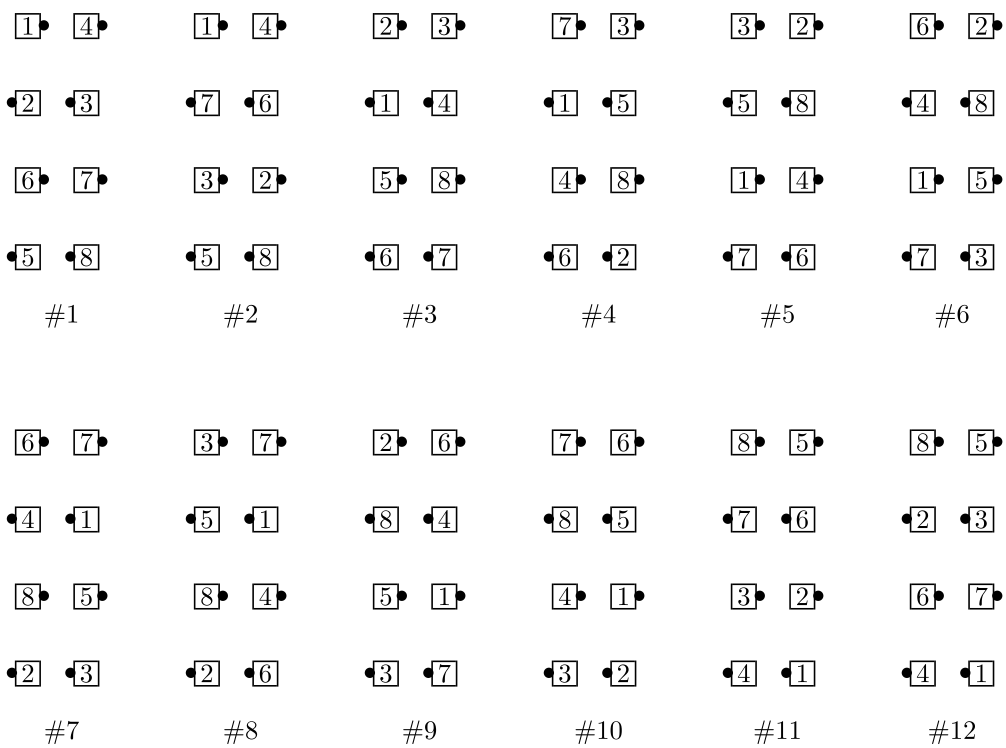
Dancer #1 cannot move to position b or g in this vertical orientation (he cannot choose his "own" position or his "opposite's" position since doing so would cause the call to have an order divisible by 4, while our desired order is only 2). Then, there are exactly six positions from which dancer #1 can choose.

Suppose dancer #1 fills position a. Then dancer #8 fills position h since these two dancers are opposites. Also, there are two more dancers whose positions are determined by dancer #1's initial placement. Before the rotation, dancer #1 would have moved to #5's position, since #5 and a are corresponding positions with the rotation. Since #1's corresponding position in the vertical orientation is b, dancer b will move to position a from the "half step" formation to the "after" formation (without the rotation), or to the corresponding position #4 in that "after" formation. Then dancer #4 must occupy position b and dancer #5 must occupy position g since these two dancers are opposites of each other.

Now four of our dancers in the "half step" are determined by dancer #1's position, which leaves only four dancers to find a position within the vertical orientation. These four dancers are #2, #3, #6, and #7, and the remaining spots to fill are c, d, e, and f. Similarly as before, dancer #2 cannot move to position d or e, since this would result in an order that is divisible by 4. Then there are exactly two positions that dancer #2 can fill: c or f. Suppose dancer #2 moves to position c. Then since #7 is #2's opposite, he must move to position f, which is c's opposite. Before the rotation, dancer #2 would have moved to dancer #6's position, since #6 and c are corresponding rotated positions. Since dancer #2's corresponding position is d, dancer d will move to position c

without the rotation, and to corresponding position #3 with the rotation. So dancer #3 must occupy position d, while #6 takes on position e. Thus the remaining positions are determined by dancer #2's initial placement.

Now all dancers have their positions in the "half step". Since dancer #1 has six choices of position, dancer #2 has two choices of position, and all other dancers' positions are determined by these two dancers, there are a total of $6 \cdot 2 = 12$ different "half steps". Below are the twelve possible "half steps", including the one already illustrated.



It is interesting to note that all of the twelve "half steps" are a result of two disjoint 4-cycles (each consisting of two pairs of opposites). As we have previously seen, these cycles have order 2; but, it is rather curious that these disjoint 4-cycles are the only possible partitioning of dancers that result in order 2 with rotation.

Thus if no dancers are fixed, a call may have order 2, 4, 8, or 12.

In General:

The cases for order with rotation are summarized below. Notice that with this rotation in formation, the only possible orders of a call are 2, 4, 8, and 12, and that if four or more dancers are fixed, the order must be 4.

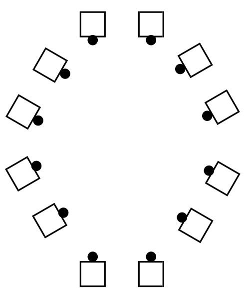
Number of Fixed Dancers	Possible Orders of a Call
8	4
6	4
4	4
2	4, 12
0	2, 4, 8, 12

It may appear that the order 16 should exist, since the maximum order without rotation is 8 and we have the potential to double this order by adding "half steps". But, in fact, it is not possible to have order 16. We can rule out this possibility since it could only result from an 8-cycle. In this case, the rotations would be complete at the end of the 8-cycle, resulting in order 8, not 16. The maximum order for these types of calls, then, is 12.

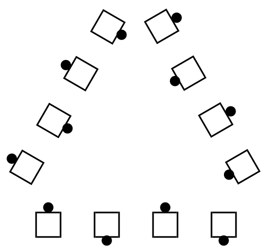
Note that other orders without the rotation can be doubled (by adding "half steps"). All of these appear in the table, except for 6 (which would result from an order 3 call without the rotation). This is particularly interesting since these 3-cycles (without rotation) require twelve iterations (with rotation) to return to the starting arrangement, and completing only six iterations of the call would result in each dancer ending in his/her opposite's position.

4 Square Dance Variations - Hexagon Dancing

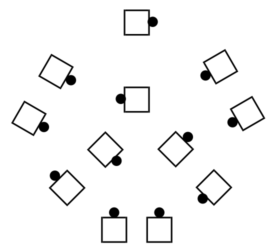
There is a relatively new variation of square dancing that involves a slight change in the definitions of calls and, perhaps, in the "rules" of the dance as well. This variation is called hexagon dancing (or "hex" dancing), and instead of using eight dancers, it uses twelve. Every square dance call can be performed in a hexagon (with some modifications as explained below), and all formations can be seen in a hexagon as well (although they will have an additional four dancers). Below are a few of the hexagon formations (their equivalent square formations can be seen in section 1).



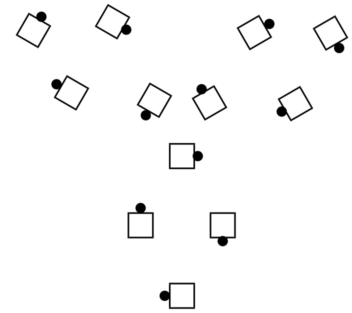
hexed set



right-handed waves

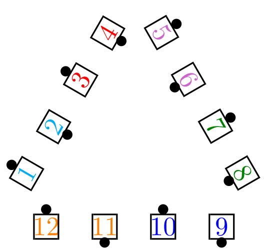


quarter tag

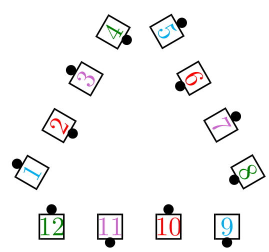


diamonds

Many of the fundamental rules of square dancing carry over to hex dancing; for instance, partners are formed the same way. Other rules are different; for example, while dancers still need to be aware of their opposite, now each dancer has two opposites, and accordingly, calls will preserve hex dance symmetry. These relationships are shown in the diagrams below, using the equivalent formation of right-handed "parallel" waves in a hexagon.



dancers with their partners



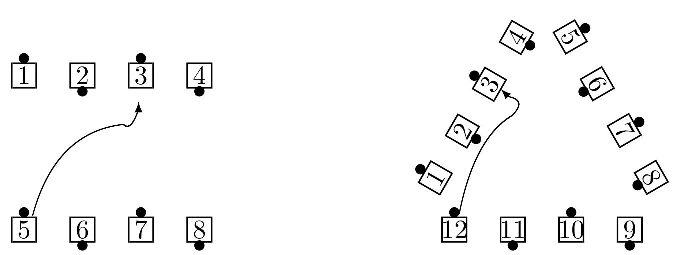
dancers with their opposites

4.1 Hex Dancing Modifications

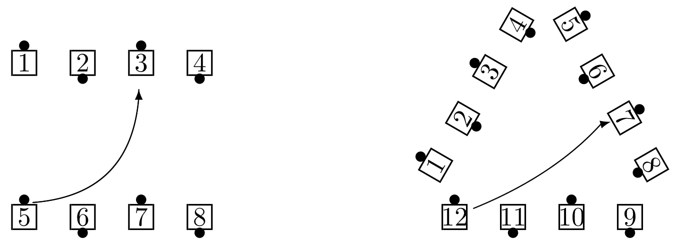
In hex dancing, each square dancing call is modified, although many fundamental parts of the call remain the same. Dancers not only need to know the definition of a call, but need to be able to figure out which other dancers are participating in the call as well. In a four-couple square, dancers learn to complete the calls with the correct amount of spinning and rotating. This same amount will not work for some calls in hexagons because the dancers may not be lined up in the same formation (since there are four extra dancers). In this case, the call may need to be fractionalized. For example, in a four-couple square one particular call may require two dancers to switch places, while in a hexagon, the same call would require three dancers to switch places. In this case, the call would be fractionalized, and instead of switching places (or rotating 180°), the three dancers would only rotate $\frac{2}{3}$ of the normal amount and thus would rotate only 120°.

This same rule is followed in many cases. All calls involving a rotation at the center of the hexagon should multiply the original angle of rotation by $\frac{2}{3}$. Then, if a call requires a dancer to cross the center of the hexagon, the dancer will rotate only $\frac{2}{3}$ the amount in the definition of the call for square dancing. To remember this rule, dancers will say that they should "under-achieve", always doing less work (around the center) than they would have in a square.

It is important when finding the hex dancing equivalent of a call to carefully follow the definition of the call, modifying angles around the center as necessary. The traffic pattern of a call will determine the ending positions of the dancers. Moving around the center in a clockwise direction versus a counterclockwise direction will change the final arrangement of dancers in a hexagon, but not in a square, so it is vital for dancers to know the exact definition of a call and follow through with it precisely. To see this, consider one dancer moving either clockwise or counterclockwise around the center of the formation to arrive at their end position.



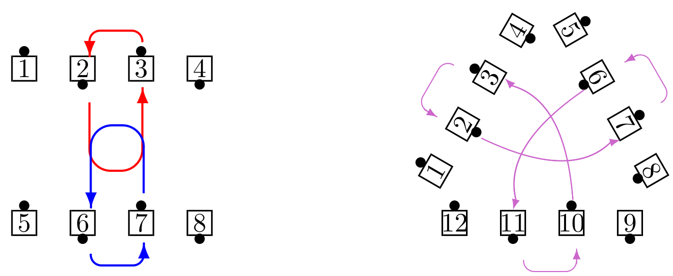
moving clockwise around the center



moving counterclockwise around the center

We see that the direction a dancer moves around the center of the formation can greatly affect the ending position of that dancer in a hexagon, since a dancer now has an extra wave to consider, but that in a square, the ending position of this dancer is the same regardless of which direction the dancer moves around the center.

The following diagram shows how the traffic pattern of a call in square dancing determines the traffic pattern for the call in hex dancing as well. Remember that the definition of the call in both types of dancing is the same, but the hex dancing version of the call has a slight modification in angles around the center of the formation. The call shown below is called *Centers Scoot Back*.



As hexagon dancing is a fairly new idea in square dancing, many dancers are not aware of this variation and articles discussing the nature of hex dancing are sparse. However, there are a few excellent resources on hex dancing and any caller or dancer interested in this variation should first look to Clark Baker's paper, Bill Eyler's paper, or any of several animation websites which include an illustration of calls in a hexagon and in a square for comparison ([1], [5], [7], [12]).

4.2 A Hex Dancing Group

Now that we have a method for dancing a call in a hexagon formation, we can convert all calls, modifying their definitions, into the appropriate call in a hexagon (by changing the angles as necessary). We can create a function that does this. Our function will change square dancing calls to hexagon dancing calls by leaving parts of the call that do not involve the center of the formation the same, and modifying parts of the call involving the center to rotating only $\frac{2}{3}$ the original amount. We want to pass every call through our function, so that we are left with only hexagon dancing calls.

Let $h : \{\text{square dancing calls}\} \to \{\text{hex dancing calls}\}$ be the function described above and let c be a square dancing call. Then $h(c)$ will be the conversion of c to its corresponding call in a hexagon.

Once we have converted all formations and calls into their hexagon equivalents using the function h, we can formalize a hexagon dancing group in the same way as for squares, now using $h(c)$ instead of c as our calls. The traffic pattern of a call is used specifically in the conversion of the call from a square dancing movement to a hexagon dancing movement. Once all calls are converted, we can "forget" the traffic patterns (with our equivalence relation from section 2), and consider the hexagon dancing group relative to a formation $\mathcal{F}_{\mathcal{H}}$ (the hexagon equivalent of formation $\mathcal{F}$). This will split our list of calls into equivalence classes depending on their starting and ending arrangements of dancers, and we can look at each of these classes separately when dealing with order.

5 Possible Orders of a Square Dance Call in Hex

Similarly as in a square, we can look at the possible orders of a hex dancing call that begins and ends in the same formation (so the call may be performed multiple times) and without any rotation in formation. A hex dancing call will ultimately have the same traffic patterns as the call performed in a square, but with modifications in the angles around the center. Then the same call in a hex will result in each dancer ending either in the expected end position from the square, or in one of the opposite positions (now there are 2 opposites). Thus the order of a call in hex dancing must be less than or equal to three times the order of the same call in a square. Rearranging this equation, we have a new theorem:

Theorem 8

If N is the order of a square dance call c and H is the order of $h(c)$ in a hexagon, then $\frac{H}{N} \leq 3$.

This will help us to find the possible orders of a hex dance call, since it narrows down the possibilites to a much smaller list of orders. We will only consider the values of H up until three times the square-order.

Also, just as we could use any formation in the square-order sections (but used right-handed parallel waves for simplicity), we can use any formation in a hexagon as well, as long as we are careful around the center of the formation and modify the angles of the call accordingly. Again, we will mainly focus on right-handed waves (and one other similar formation) for simplicity.

Just as before, we can use our definition for an n-cycle of dancers, and the order of an n-cycle is still n in hexes. However, in hex dancing, since there are twelve dancers, we can think of having n-cycles where $n = 1, 2, \cdots, 12$. Again, 1-cycles represent fixed dancers.

We see that if n is a multiple of three, the n-cycle is possible since it could include a dancer and his two opposites. If n is not a multiple of three, it is only possible to have n-cycles if there are $3k$ of them (since the n-cycle cannot consist entirely of full sets of opposites), where $k = 1, 2, 3, 4$ (since there are only twelve dancers). For example, since two is not a multiple of three, we can only have 2-cycles if we have three 2-cycles or six 2-cycles, so as to follow the rules of hex dance symmetry. (Note that we cannot have nine disjoint 2-cycles since this would involve eighteen dancers, and we only have twelve.) For this reason, we cannot have 5-cycles, 7-cycles, 8-cycles, 10-cycles, and 11-cycles, since we are limited by how many dancers are in a hex.

Also note that Theorem 3 is still valid in a hex, so we can use the cycles described above to help determine the order of a call.

Since the maximum possible order in a square (without rotation) is 8, the maximum possible order for a hexagon is $3 \cdot 8 = 24$, but this maximum can be lowered as well. In fact, we can cut this maximum in half, using a similar argument as in the square.

Theorem 9

The maximum order of a hexagon dancing call starting and ending in $\mathcal{F}_{\mathcal{H}}$ (preserving hex dance symmetry) is 12.

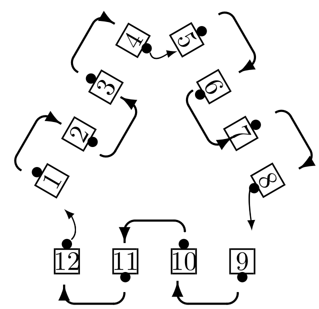
Proof

Using the same methods as before, we see that there are only eight ways to partition the twelve dancers into disjoint cycles (ignoring 1-cycles):

- one 12-cycle
- one 9-cycle and one 3-cycle
- two 6-cycles
- one 6-cycle and two 3-cycles
- one 6-cycle and three 2-cycles
- three 4-cycles
- four 3-cycles
- six 2-cycles

By Theorem 3, we see that the maximum possible order of calls with the cycles above is 12. Note that it is permissible to ignore the possibility of 1-cycles, since this will not affect the order of a call.

Again, it is important to note that it is possible to have order 12. By imagining the right-handed wave formation as a triangle with four positions along each side of the triangle, we can simply ask the dancers to move clockwise to the next position in the triangle (completing an *Inroll Circulate* for example), guaranteeing that they will end in all positions before arriving back to their starting positions, giving an order of 12.



Thus the maximum order of a hex dance call is 12. □

<u>Remark:</u> The proof of Theorem 9, in light of Theorem 3, shows that the only possible hex orders are 1, 2, 3, 4, 6, 9, and 12.

We can classify these hex-orders by looking at each square-order individually. Paying special attention to the square-order, we will look at the cases when a call c has square-order $N = 1$, 2, 3, 4, 6, and 8, and determine which hex-orders are possible for $h(c)$ in each of these cases.

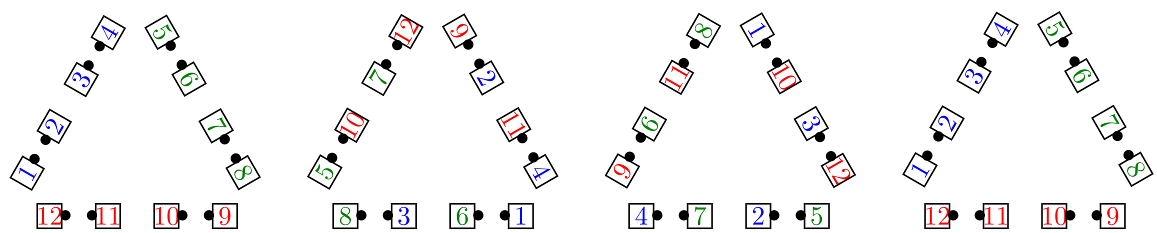
<u>Case 1:</u> $(N = 1)$

When $N = 1$, by Theorem 8, $H = 1$, 2, or 3. In other words, if c is a call that has order 1 in a square, then the hex-order of $h(c)$ is limited to 1, 2, or 3. We must consider each of these orders in a hexagon and see whether each order is possible. Keep in mind in each of these cases that the order of the same call in a square is 1, resulting from dancers starting and ending in the same positions of the formation.

$H = 1$: This call is the identity call of our hex dancing group. Simply put, dancers should just not move. They will start and end in the same position in both the square and in the hexagon.

$H = 2$: This order is not possible with $N = 1$, since at least one dancer would end in one of the opposites' positions, resulting in a 3-cycle in the hexagon. Then by Theorem 3, H would be divisible by 3. Not possible!

$H = 3$: This type of call is possible. For example, an *Eight Chain Thru*, starting and ending in an "Eight Chain Thru" formation, has order 1 in a square, but order 3 in a hexagon. This call (in a square) has each dancer follow the shape of the formation, occupying each position during the call, and ending in their original starting positions. In a hexagon, the ending positions are shown below:



before | after 1 iteration | after 2 iterations | after 3 iterations

Thus if $N = 1$, then $H = 1$ or $H = 3$.

Case 2: ($N = 2$)

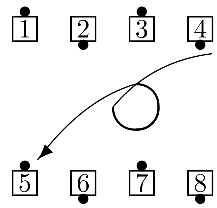
When $N = 2$, by Theorem 8 and the Remark after Theorem 9, $H = 1$, 2, 3, 4, or 6. By Theorem 3, $N = 2$ means that a call c in a square involves only 2-cycles and 1-cycles. (In order for c to have order 2, there must be at least one 2-cycle.) As argued in the $N = 1$ case, a 1-cycle in the square could result in either a 1-cycle or a 3-cycle in the hex.

If a 2-cycle consists of a pair of opposites, then in $h(c)$, the corresponding three dancers in the hex must end up in these same three positions. These three dancers can be arranged in such a way that they form either three 1-cycles or one 3-cycle.

On the other hand, if a 2-cycle (in the square) consists of a pair of non-opposites, then there must be another 2-cycle containing the opposites of the dancers in the first 2-cycle (in order to preserve square dance symmetry). In $h(c)$, the corresponding six dancers must each move (symmetrically) to a non-opposite's position, forming either three 2-cycles or one 6-cycle in the hex.

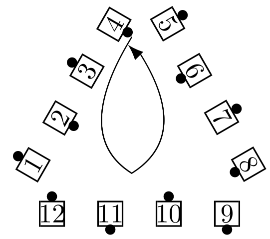
Combining all of these possibilities using Theorem 3, if c has order 2, then $h(c)$ either has order 1, 2, 3, or 6.

$H = 1$: This order is possible, so long as a dancer in the square trades positions with his/her opposite, going around the center one full rotation. To visualize this, we will look only at dancer #4, knowing that dancer #4's opposite(s) will be completing the same motion:



#4 and #5 trade

(360°)



#4 remains in position

(240°)

Note: Assuming $H = 1$, if the dancers go around the center of the hexagon at least once, there are two possibilities. If the dancers go around an even number of times, then this will result from dancers ending in their own positions in the square. Otherwise, dancers must have ended in their opposites' positions of the square. If dancers do not go around the center of the hexagon, then these dancers are not subject to the $\frac{2}{3}$ angle rule, which causes the change in ending position after interaction with the center of the hexagon. In this case, $N = 1$. Thus if $H = 1$, the order in a square is either 1 or 2, and so we need not consider $H = 1$ after this point.

$H = 2$: An example of a call that remains order 2 in a hex, using right-handed waves, is *Trade*, meaning that each dancer should switch places with their partner (or the person to his/her right in right-handed waves). The order does not change since each dancer works directly with the person next to them, and will continue to work with that same person if *Trade* is called again.

$H = 3$: This order can result from a dancer trading places with his/her opposite (without a full rotation around the center of the formation). In a square, this has order 2 since each dancer has one other opposite. In a hex, this has order 3 since each dancer has two other opposites.

$H = 6$: An example of this type of call is *Centers Scoot Back.* In a square, the dancers on the end of the wave do nothing, and the dancers in the center of the wave participate in the call by trading places with their center partner. The dancer who is heading towards the center of the hexagon should go around the center before going back to his center partner's position. Diagrams for this call can be seen in the previous section.

Thus if $N = 2$, then $H = 1$, 2, 3, or 6.

Case 3: ($N = 3$)

When $N = 3$, by Theorem 8 and the Remark after Theorem 9, $H = 1$, 2, 3, 4, 6, or 9. In the previous case, we showed $H = 1$ only when $N = 1$ or $N = 2$, so we will ignore $H = 1$ for the remaining cases. Recall that $N = 3$ is possible if and only if a call c consists of two disjoint 3-cycles and two fixed dancers (opposites).

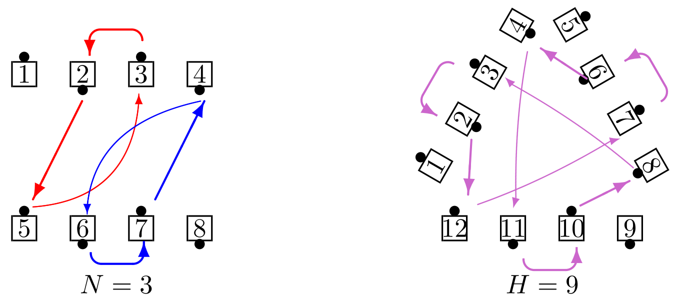
$H = 2$: This is impossible. Since a 3-cycle in a square cannot contain both a dancer and his/her opposite, the call c must actually consist of two disjoint 3-cycles. This would require $h(c)$ to use nine people in a hexagon, whether these nine dancers are in one cycle or many. The order in a hex cannot be 2 since these nine dancers cannot form pairs.

$H = 3$: It is possible for $N = H = 3$. For example, a call c that consists of two fixed dancers and two 3-cycles, each contained in a separate wave and not involving rotation about the center of the square, will have the same order in a square as $h(c)$ has in a hex because $h(c)$ will consist of three fixed dancers and three 3-cycles (one in each wave).

$H = 4$: Similarly as the case for $H = 2$, this case is impossible since the three hex dancers corresponding to the fixed dancers in the square must remain fixed (otherwise they would form a 3-cycle, which is incompatible with $H = 4$), and the remaining nine dancers cannot form cycles of length 4 (leaving one person out would violate the hex dance symmetry).

$H = 6$: It may seem that $H = 6$ is possible, but it is not. This is because the nine hex dancers corresponding to the two 3-cycles in the square cannot be split into one 6-cycle and one 3-cycle. Doing so would violate the hex dance symmetry and have opposites doing different things throughout the call.

$H = 9$: Consider the call below which has $N = 3$ and $H = 9$.

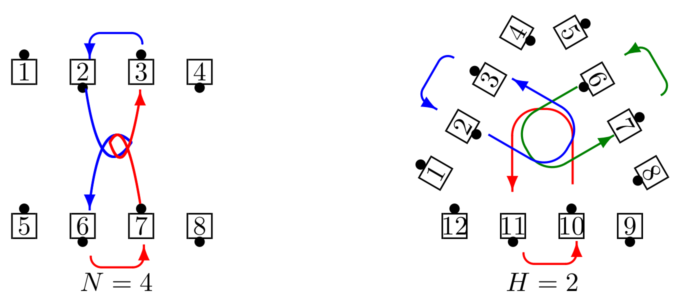


Thus if $N = 3$, then $H = 3$ or $H = 9$.

<u>Case 4:</u> $(N = 4)$

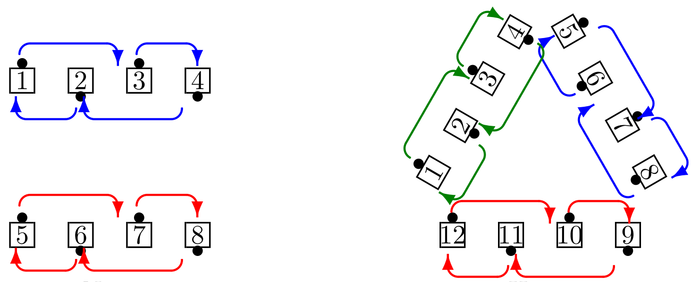
When $N = 4$, by Theorem 8 and the Remark after Theorem 9, $H = 2$, 3, 4, 6, 9, or 12. Recall that $N = 4$ is possible only if the call consists of at least one 4-cycle.

$H = 2$: This type of call is possible, resulting from a 4-cycle in which two of the dancers go a full rotation around the center of the square. When the call is converted to its equivalent in a hexagon, this full rotation will be cut short and the dancer will end up in his center partner's place.



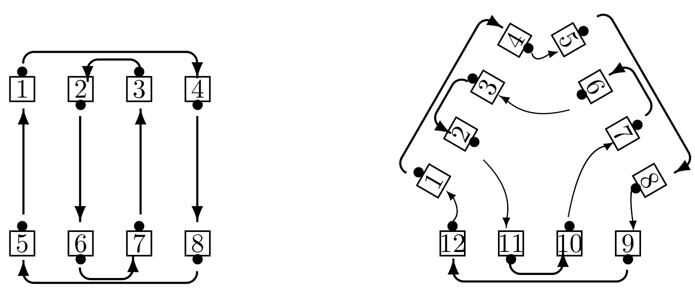
$H = 3$: Thinking backwards, we will consider the case when $H = 3$. By Theorem 3, the twelve hex dancers must be split between at least one 3-cycle and possibly 1-cycles. The 1-cycles could only have come from 1-cycles or 2-cycles in a square, depending on the amount of rotation at the center of the formation. For the 3-cycles, if they consist of complete sets of opposites, then the corresponding square-order is either 1 or 2. Otherwise, the corresponding square-order is 3 or 6, again depending on the amount of rotation around the center of the formation. In any case, by Theorem 3 for a square, it is not possible for $N = 4$, and so we have a contradiction.

$H = 4$: Calls that keep dancers on their own side of the square (not requiring them to cross the center) will remain to have order 4 in the hexagon. For example, calls like *Swing Thru* that allow dancers to work only with the other dancers in their wave will have order 4, since these dancers will continue to work with each other in the hex version of the call.



$N = 4$ $\qquad$ $H = 4$

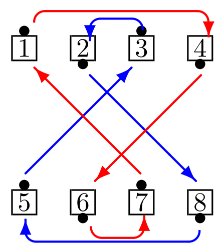
$H = 6$: The following is a call that takes two disjoint 4-cycles in a square that, when performed in a hexagon, become two disjoint 6-cycles. This particular call is named *All-8 Circulate*.



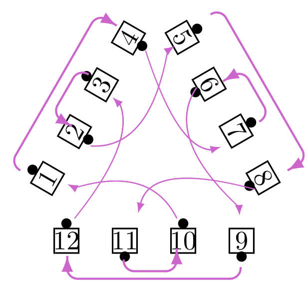
$N = 4$ $\qquad$ $H = 6$

$H = 9$: If $H = 9$, then there are two possibilities: one 9-cycle and one 3-cycle, or one 9-cycle and three fixed dancers. In either case, the 3-cycle or the three fixed dancers must consist of a complete set of opposites. Then the corresponding pair of opposites in the square cannot be a part of a 4-cycle, which leaves six dancers to choose from in the square to form the 4-cycle required for $N = 4$. We see that after the 4-cycle is formed, we will be left with an additional pair of opposites. In the hex, this means there are an additional three dancers (on top of the original 3-cycle or three fixed dancers) that are not a part of our original 9-cycle, and thus we have a contradiction. Therefore, it is not possible in this case for $H = 9$.

$H = 12$: An example of a call with $H = 12$ is called *Bias Circulate*. This call consists of dancers performing a certain movement depending on which position they currently occupy. They end up completing a pattern: inside, inside, outside, outside (in reference to which position they will occupy next, the inside or outside of a wave).



$N = 4$



$H = 12$

Thus if $N = 4$, then $H = 2$, 4, 6, or 12.

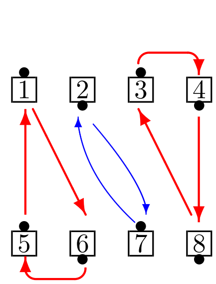
Case 5: ($N = 6$)

When $N = 6$, by Theorem 8 and the Remark after Theorem 9, $H = 2$, 3, 4, 6, 9, or 12.

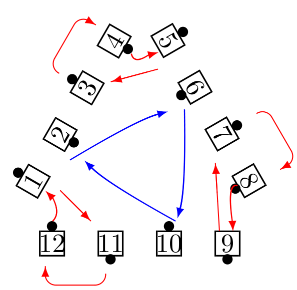
This case follows from the previous cases. It is important to remember that if $N = 6$, there are only two possibilities in a square: either the eight dancers form two 3-cycles, leaving one dancer to trade places with his/her opposite, or there is one 6-cycle, leaving two dancers fixed or trading places with each other.

If the former is the case, our 3-cycles in the square will have mini-order 3 or 9 in the hex by Case 3. The other two dancers in the square are now three dancers in the hex because each dancer now has two opposites, and so the mini-order for these dancers is either 1 or 3 in the hex by Case 2. Then by Theorem 3, the order of this type of call is $\text{lcm}(3, 1 \text{ or } 3) = 3$ or $\text{lcm}(9, 1 \text{ or } 3) = 9$.

Here is an example of a call having two 3-cycles and a pair of dancers who trade:



$N = 6$



$H = 3$

If we have the other type of call which has one 6-cycle and one pair of fixed or trading dancers in the square, then our two fixed or trading dancers become three dancers in the hex, leaving nine hex dancers left to account for. In this case we can still have orders 3 and 9, but it is not possible to have orders 2, 4, 6, or 12. This is because we cannot split our 9 dancers into cycles of length 2, 4, or 6 without violating the hex dance symmetry. We cannot have only 2-cycles and 4-cycles because they would leave one dancer out. We cannot have 6-cycles either because the only way this would be possible is if we split our nine dancers into a 6-cycle and either one 3-cycle or three 1-cycles. The three dancers not in the 6-cycle must be opposites, which means the 6-cycle (in the hex) must consist of two complete sets of opposites. In the square, these two sets of opposites would only be four dancers involved in our original square 6-cycle. This is a contradiction; thus it is impossible to partition our nine dancers to obtain a 6-cycle when $N = 6$. Also, since only 9 dancers are participating in the call, we cannot have order greater than 9, using logic similar to the proof of Theorem 9.

Thus if $N = 6$, $H = 3$ or $H = 9$.

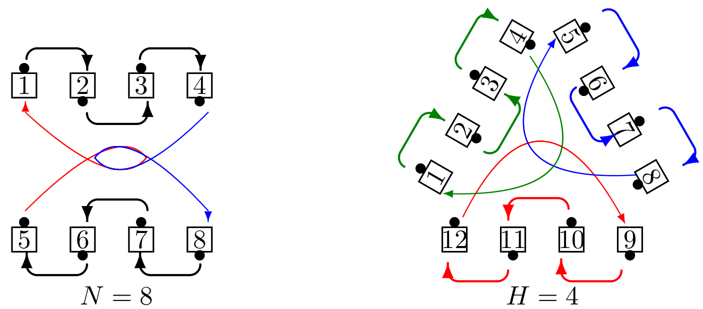
Case 6: ($N = 8$)

When $N = 8$, by Theorem 8 and the Remark after Theorem 9, $H = 2$, 3, 4, 6, 9, or 12.

$H = 2$: Thinking backwards, we will consider the case when $H = 2$. In fact, this can only be achieved from a call of order 2 or 4 in a square. We see that if $H = 2$, by Theorem 3, our twelve dancers are split into disjoint 2-cycles and 1-cycles (there must be at least one 2-cycle for $H = 2$). We have shown that a 1-cycle in a hex can result from a 2-cycle or a 1-cycle in a square (either with a full rotation at the center or without). Similarly, a 2-cycle in a hex comes from either a 2-cycle in a square (without a full rotation or with an odd number of full rotations), or a 4-cycle in a square (with an even number of full rotations). Then by Theorem 3, the order in a square is either 2 or 4. Thus it is not possible for $H = 2$ when $N = 8$.

$H = 3$: Using the same argument as in Case 4, we see that this case is impossible. By Theorem 3, we see that the only possible square-orders giving $H = 3$ are $N = 1$, 2, 3, or 6.

$H = 4$: An example of a call with $N = 8$ and $H = 4$ is a variation of *Inroll Circulate* (illustrated in the proof of Theorem 4), where two of the dancers go a full rotation around the center.

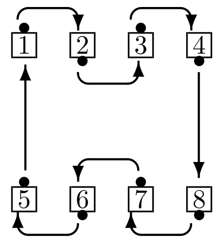


$H = 6$: A 6-cycle in a hexagon must involve two groups of opposites (to preserve the hex dance symmetry). But since there is only one opposite for each dancer in a square, the most dancers used for the corresponding cycle in a square is 4, not 8.

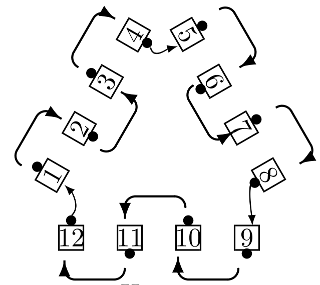
The only other way to arrive at an order of 6 in a hexagon is to have two 3-cycles and three 2-cycles, or to have one 3-cycle, three 2-cycles, and three fixed dancers. For the 3-cycles, we have already found that the potential orders in a square are 1, 2, 3, or 6; for the 2-cycles, possible square-orders are 2 or 4; and for the 1-cycles, the possible square-orders are 1 or 2. However, not all combinations of these orders are possible, since they depend on how many other dancers are participating in the call. In any case, lcm ≤ 6 in a square, so we can never have $N = 8$ in this situation. Thus this case is not possible.

$H = 9$: A 9-cycle in a hexagon would require the other three dancers to complete a different movement, either becoming part of a 3-cycle or remaining fixed. In order to preserve hex dance symmetry, these three dancers must be opposites. Then either the two corresponding dancers are fixed in the square, so the order cannot be 8; or, the opposites are trading places, in which case the square-order is determined by the other six dancers and cannot be 8 either. Thus it is not possible for $N = 8$ and $H = 9$ simultaneously.

$H = 12$: A call that has $N = 8$ and $H = 12$ is *Inroll Circulate*.



$N = 8$



$H = 12$

Thus if $N = 8$, $H = 4$ or $H = 12$.

In General:

We organize the results from the previous cases below.

Order of c in a Square	Possible Orders of $h(c)$ in a Hex
1	1, 3
2	1*, 2, 3, 6
3	3, 9
4	2*, 4, 6, 12
6	3, 9
8	4*, 12

We see that every order less than 12 shows up in the table, except for 5, 7, 8, 10, and 11, which we already knew to be impossible orders for a hex dancing call. Also, we can see that each hex-order is either $\frac{1}{2}$, 1, $\frac{3}{2}$, or 3 times the square-order. The starred orders refer to those calls which require a full rotation around the center of the formation.

6 Concluding Remarks

We have observed that we can consider square dancing from the viewpoint of group theory in mathematics. Since we can form groups of square dancing calls, we can use various concepts from group theory to look into the different aspects of a square dancing call, specifically the order of a call. We discover that the traffic pattern of a square call plays a large role in determining the order of the corresponding hex call, especially when dealing with the extra dancers of a hexagon and the distortion of angles in the formation. We have analyzed the possible orders of a non-formation-changing call in squares, both with rotation in formation and without, as well as the orders of a call in a hexagon without any added rotation in the formation.

Future areas of study may include the possibility of rotating a formation in a hexagon and the impact that other formations (instead of right-handed waves) have on the hexagon orders. In addition to hexagon dancing, there are other polygon formations, like octagon dancing, which involves sixteen dancers and alters angles at the center by $\frac{1}{2}$ instead of the $\frac{2}{3}$ for hexagons. It would be interesting to see how the order of a call changes between the square, hexagon, and octagon, as well as other polygon formations. Another area of study lies in determining whether the possible orders of a square (or hex) dancing call can be obtained from a single call (rather than a queue of calls). If there are orders that cannot be obtained from a single call, then perhaps callers (or dancers) may wish to create new calls to fill in these gaps.

It may be illuminating for square dancers to understand the mathematical concepts that exist in square dancing, as it adds an interesting factor to the dance and turns square dancing into an even more mentally engaging activity than it already is considered. But even more importantly, square dance callers can appropriately use such ideas, particularly that of the order of a call. By understanding what it means for a call to have an order, callers may "spice up" the dance. Instead of asking dancers to complete a particular call, a caller may choose a call of the same order (so long as the permutations of dancers are the same), moving dancers to the same ending positions, but possibly in a more exciting manner, by having the dancers interact more with each other throughout the call or by requiring dancers to perform some of the more interesting, but less common calls. The square dance group really benefits both the caller and the dancers. The caller may have more options as to what calls he can ask dancers to perform from various formations and arrangements of dancers, while the dancers, because of this, will be able to have more fun during the dance.

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